Computationally Efficient GES Cascade Observer for Attitude Estimation

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Abstract— **This paper presents the design, analysis, and performance evaluation of a novel cascade observer for attitude estimation. First, a sensor-based observer, which resorts to rate gyro readings and a set of vector observations, estimates the rate gyro bias. Afterwards, a second observer explicitly estimates the attitude in the form of a rotation matrix based on the rate gyro measurements, the vector observations, and the estimate of the rate gyro bias provided by the first observer. The error dynamics of the overall cascade estimation system are globally exponentially stable (GES) and do not suffer from drawbacks common to attitude estimation solutions such as singularities, unwinding phenomena, or topological limitations for achieving global asymptotic stability (GAS). In addition, the proposed system is computationally efficient and hence it is easily implementable with low computational capabilities. The fact that the observer does not evolve explicitly on $SO(3)$, providing in fact estimates that converge asymptotically to $SO(3)$, is also addressed and an effective and efficient solution is proposed. Finally, the resulting estimator is evaluated, where a Motion Rate Table (MRT) that provides ground truth data is employed for performance evaluation purposes.**

I. Introduction

Attitude estimation has been a hot topic of research in recent years, as evidenced by the large number of publications on the subject, see e.g. [1], [2], and [3]. The Extended Kalman Filter (EKF) and variants have been at the core of numerous stochastic solutions, see e.g. [4], [5], [6], and [7], while nonlinear alternatives, aiming for stability and convergence properties, have been proposed in [8], [9], [10], [11], and [12], to mention just a few, see [13] for a thorough survey on the subject of attitude estimation.

Lack of convergence guarantees, singularities, unwinding phenomena and topological limitations for achieving global asymptotic stability are common drawbacks of attitude estimation solutions. In previous work by the authors, [14], a sensor-based attitude estimation filter was derived that has globally asymptotically stable (GAS) error dynamics and does not carry any of the aforementioned limitations. Unfortunately, that solution is computational expensive. Indeed, it requires the solution of a matrix differential Riccati equation associated to a state of dimension $3\,(N+1)$, where N is the number of vector observations. In addition, the final rotation matrix is obtained from the solution of the Wahba's problem, which involves, in general, an SVD problem.

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The main contribution of this paper is the design, analysis, and performance evaluation of a novel cascade attitude observer that: i) has globally exponentially stable (GES) error dynamics; ii) is computationally efficient; iii) is based on the exact angular motion kinematics; iv) builds on well-established Lyapunov results; v) explicitly estimates rate gyro bias and copes well with slowly time-varying bias; and vi) has a complementary structure, fusing low bandwidth vector observations with high bandwidth rate gyro measurements. In this paper, the sensor measurements are included directly in the system dynamics, following the approach introduced in [14], and the kinematics are propagated using the angular velocity provided by a three-axis rate gyro, whose bias is also considered. A novel computationally efficient observer is designed for this system, that yields an estimate of the rate gyro bias, and that feeds a second novel observer for the rotation matrix, which is also computationally efficient. The overall closed-loop error dynamics are shown to be GES and the estimates of the rotation matrix converge asymptotically to the Special Orthogonal Group, $SO(3)$. An additional solution refinement is provided that yields solutions arbitrarily close to $SO(3)$, keeping at the same time low computational requirements. Finally, the proposed solution does not exhibit any of the aforementioned drawbacks common to attitude estimation solutions such as singularities, unwinding phenomena, or topological limitations for achieving global asymptotic stabilization on $SO(3)$, see [15].

The paper is organized as follows. The problem statement is introduced in Section II, whereas the observer design and stability analysis are presented in Section III. Experimental results are provided and discussed in Section IV and finally, in Section V, the main contributions and conclusions of the paper are summarized.

A. Notation

Throughout the paper the symbol $\mathbf{0}$ denotes a matrix (or vector) of zeros and $\mathbf{I}$ an identity matrix, both of appropriate dimensions. A block diagonal matrix is represented as $\operatorname{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_\mathrm{n})$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, $\mathbf{x} \times \mathbf{y}$ represents the cross product.

II. Problem statement

Let $\{I\}$ be an inertial reference frame, $\{B\}$ a body-fixed reference frame, and $\mathbf{R}(t) \in SO(3)$ the rotation matrix from $\{B\}$ to $\{I\}$. The attitude kinematics, expressed in the form of a rotation matrix, are given by $\dot{\mathbf{R}}(t) = \mathbf{R}(t)\mathbf{S}\left(\boldsymbol{\omega}(t)\right)$, where $\boldsymbol{\omega}(t) \in \mathbb{R}^3$ is the angular velocity of $\{B\}$, expressed in $\{B\}$, and $\mathbf{S}(.)$ is the skew-symmetric matrix

$$\mathbf{S}(\boldsymbol{x}) := \begin{bmatrix} 0 & -\boldsymbol{x}_z & \boldsymbol{x}_y \\ \boldsymbol{x}_z & 0 & -\boldsymbol{x}_x \\ -\boldsymbol{x}_y & \boldsymbol{x}_x & 0 \end{bmatrix},\ \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_x \\ \boldsymbol{x}_y \\ \boldsymbol{x}_z \end{bmatrix} \in \mathbb{R}^3.$$

The angular velocity is assumed to be a continuous bounded signal. Suppose that rate gyro measurements are available, corrupted by a constant bias, as given by

$$\boldsymbol{\omega}_m(t) = \boldsymbol{\omega}(t) + \mathbf{b}_{\boldsymbol{\omega}}(t), \tag{1}$$

where $\mathbf{b}_{\boldsymbol{\omega}}(t) \in \mathbb{R}^3$ is the rate gyro bias, which satisfies $\dot{\mathbf{b}}_{\boldsymbol{\omega}}(t) = 0$. In addition to the rate gyro readings, suppose that a set of N vector observations $\{\mathbf{v}_i(t) \in \mathbb{R}^3, i = 1, \ldots, N\}$ is available, in body-fixed coordinates, of known constant vectors in inertial coordinates,

$$\mathbf{r}_i = \mathbf{R}(t)\mathbf{v}_i(t),\ i = 1, \ldots, N. \tag{2}$$

In the remainder of the paper the following assumption is made:

Assumption 1: There exist at least two non-collinear reference vectors, i.e., there exist i and j such that $\mathbf{r}_i \times \mathbf{r}_j \neq \mathbf{0}$.

This assumption is necessary for attitude estimation with constant vectors in inertial coordinates, see e.g. [11] and [14], and therefore it carries no conservativeness whatsoever.

The problem considered in the paper is the design of an observer for the rotation matrix $\mathbf{R}(t)$ and the rate gyro bias $\mathbf{b}_{\boldsymbol{\omega}}(t)$ with globally exponentially stable error dynamics.

III. Observer design and stability analysis

This section details the design of the attitude observer and the stability analysis. First, a bias observer with GES error dynamics is derived, in Section III-A, that resorts directly to the vector observations. Afterwards, an attitude observer with GES error dynamics is proposed, in Section III-B, assuming that the rate gyro bias is known. The overall cascade attitude observer is presented in Section III-C, where it is shown that the resulting error dynamics are GES. Finally, refinements of the final solution are discussed in Section III-D.

A. Bias observer

The set of states of the bias observer proposed in this section corresponds to the set of vector observations, in addition to the rate gyro bias. The time derivative of the vector observations is given by

$$\dot{\mathbf{v}}_i(t) = -\mathbf{S}(\boldsymbol{\omega}(t))\,\mathbf{v}_i(t),\ i = 1, \ldots, N. \tag{3}$$

From (1) it is possible to rewrite (3) as

$$\begin{aligned}\dot{\mathbf{v}}_i(t) &= -\mathbf{S}(\boldsymbol{\omega}_m(t))\,\mathbf{v}_i(t) + \mathbf{S}(\mathbf{b}_{\boldsymbol{\omega}}(t))\,\mathbf{v}_i(t) \\ &= -\mathbf{S}(\boldsymbol{\omega}_m(t))\,\mathbf{v}_i(t) - \mathbf{S}(\mathbf{v}_i(t))\,\mathbf{b}_{\boldsymbol{\omega}}(t),\ i = 1, \ldots, N.\end{aligned}$$

Consider the bias observer given by

$$\begin{cases}\dot{\hat{\mathbf{v}}}_1(t) = -\mathbf{S}(\boldsymbol{\omega}_m(t))\,\hat{\mathbf{v}}_1(t) - \mathbf{S}(\mathbf{v}_1(t))\,\hat{\mathbf{b}}_{\boldsymbol{\omega}}(t) + \alpha_1\tilde{\mathbf{v}}_1(t) \\ \quad\vdots \\ \dot{\hat{\mathbf{v}}}_N(t) = -\mathbf{S}(\boldsymbol{\omega}_m(t))\,\hat{\mathbf{v}}_N(t) - \mathbf{S}(\mathbf{v}_N(t))\,\hat{\mathbf{b}}_{\boldsymbol{\omega}}(t) + \alpha_N\tilde{\mathbf{v}}_N(t) \\ \dot{\hat{\mathbf{b}}}_{\boldsymbol{\omega}}(t) = \sum_{i=1}^{N}\beta_i\mathbf{S}(\mathbf{v}_i(t))\,\tilde{\mathbf{v}}_i(t)\end{cases}, \tag{4}$$

where $\tilde{\mathbf{v}}_i(t) := \mathbf{v}_i(t) - \hat{\mathbf{v}}_i(t),\ i = 1, \ldots, N$, are the errors of the vector observation estimates, available for stabilization purposes, and $\alpha_i, \beta_i,\ i = 1, \ldots, N$, are positive scalar constants. Define the bias estimation error as $\tilde{\mathbf{b}}_{\boldsymbol{\omega}}(t) := \mathbf{b}_{\boldsymbol{\omega}}(t) - \hat{\mathbf{b}}_{\boldsymbol{\omega}}(t)$. Then, it is straightforward to show that the bias observer error dynamics can be written, in compact form, as

$$\dot{\tilde{\boldsymbol{x}}}_1(t) = \boldsymbol{\mathcal{A}}_1(t)\tilde{\boldsymbol{x}}_1(t), \tag{5}$$

where $\tilde{\boldsymbol{x}}_1(t) = \left[\tilde{\mathbf{v}}_1^T(t)\ \ldots\ \tilde{\mathbf{v}}_N^T(t)\ \tilde{\mathbf{b}}_{\boldsymbol{\omega}}^T(t)\right]^T \in \mathbb{R}^{3(N+1)}$ and

$$\begin{aligned}\boldsymbol{\mathcal{A}}_1(t) = &-\mathrm{diag}\left(\alpha_1\mathbf{I} + \mathbf{S}(\boldsymbol{\omega}_{\mathrm{m}}(t)), \ldots, \alpha_{\mathrm{N}}\mathbf{I} + \mathbf{S}(\boldsymbol{\omega}_{\mathrm{m}}(t)), \mathbf{0}\right) \\ &+ \begin{bmatrix} \mathbf{0} & \ldots & \mathbf{0} & -\mathbf{S}(\mathbf{v}_1(t)) \\ \vdots & & \vdots & \vdots \\ \mathbf{0} & \ldots & \mathbf{0} & -\mathbf{S}(\mathbf{v}_N(t)) \\ -\beta_1\mathbf{S}(\mathbf{v}_1(t)) & \ldots & -\beta_N\mathbf{S}(\mathbf{v}_N(t)) & \mathbf{0} \end{bmatrix}.\end{aligned}$$

Before presenting the main result of this section, the following lemma is introduced.

Lemma 1 ([14, Lemma 1]): Let $\mathbf{f}(t) : [t_0, t_f] \subset \mathbb{R} \to \mathbb{R}^n$ be a continuous and two times continuously differentiable function on $\mathcal{I} := [t_0, t_f]$, $T := t_f - t_0 > 0$, and such that $\mathbf{f}(t_0) = \mathbf{0}$. Further assume that $\left\|\ddot{\mathbf{f}}(t)\right\| \leq C$ for all $t \in \mathcal{I}$. If there exist scalar constants $\alpha^* > 0$ and $t^* \in \mathcal{I}$ such that $\left\|\dot{\mathbf{f}}(t^*)\right\| \geq \alpha^*$, then there exist $0 < \delta^* \leq T$ and $\beta^* > 0$ such that $\|\mathbf{f}(t_0 + \delta^*)\| \geq \beta^*$.

The following theorem is the main result of this section.

Theorem 1: Under Assumption 1, consider the bias observer (4), where $\alpha_i > 0$, $\beta_i > 0$, $i = 1, \ldots, N$, are positive scalar parameters. Then, the origin of the observer error dynamics (5) is a globally exponentially stable equilibrium point.

Proof: Consider the Lyapunov function candidate

$$V_1(t) := \tilde{\boldsymbol{x}}_1^T(t)\mathbf{D}\tilde{\boldsymbol{x}}_1(t) = \frac{1}{2}\sum_{i=1}^{N}\beta_i\left\|\tilde{\mathbf{v}}_i(t)\right\|^2 + \frac{1}{2}\left\|\tilde{\mathbf{b}}_{\boldsymbol{\omega}}(t)\right\|^2,$$

where $\mathbf{D} := \frac{1}{2}\mathrm{diag}\left(\beta_1\mathbf{I}, \ldots, \beta_{\mathrm{N}}\mathbf{I}, \mathbf{I}\right)$. Clearly,

$$\gamma_1\left\|\tilde{\boldsymbol{x}}_1(t)\right\|^2 \leq V_1(t) \leq \gamma_2\left\|\tilde{\boldsymbol{x}}_1(t)\right\|^2, \tag{6}$$

where $\gamma_1 := \frac{1}{2}\min(1,\ \beta_1,\ \ldots,\ \beta_N)$ and $\gamma_2 := \frac{1}{2}\max(1,\ \beta_1,\ \ldots,\ \beta_N)$. The time derivative of $V_1(t)$ can be written as

$$\dot{V}_1(t) = -\tilde{\boldsymbol{x}}_1^T(t)\mathbf{C}_1^T\mathbf{C}_1\tilde{\boldsymbol{x}}_1(t) \leq 0, \tag{7}$$

where

$$\mathbf{C}_1 = \begin{bmatrix} \sqrt{\alpha_1\beta_1} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ldots & \mathbf{0} & \sqrt{\alpha_N\beta_N} & \end{bmatrix} \in \mathbb{R}^{3N\times 3(N+1)}.$$

It is well known that, if in addition to (6) and (7), the pair $(\boldsymbol{\mathcal{A}}_1(t), \mathbf{C}_1)$ is uniformly completely observable, then the origin of the linear time-varying system (5) is a globally exponentially stable equilibrium point, see [16, Example 8.11]. The remainder of the proof amounts to show that the pair $(\boldsymbol{\mathcal{A}}_1(t), \mathbf{C}_1)$ is uniformly completely observable. For any piecewise continuous, bounded matrix $\mathbf{K}_1(t)$, of compatible dimensions, uniform complete observability of the pair

$(\boldsymbol{\mathcal{A}}_1(t), \mathbf{C}_1)$ is equivalent to uniform complete observability of the pair $(\mathbf{A}_1(t), \mathbf{C}_1)$, with $\mathbf{A}_1(t) := \boldsymbol{\mathcal{A}}_1(t) - \mathbf{K}_1(t)\mathbf{C}_1$, see [17, Lemma 4.8.1]. Now, notice that, attending to the particular forms of $\mathbf{C}_1$ and $\boldsymbol{\mathcal{A}}_1(t)$, there exists a continuous bounded matrix $\mathbf{K}_1(t)$, which depends explicitly on the observer parameters, the rate gyro readings, $\boldsymbol{\omega}_m(t)$, and the vector observations $\mathbf{v}_i(t)$, $i = 1, \ldots, N$, such that

$$\mathbf{A}_1(t) = \begin{bmatrix} \mathbf{0} & \ldots & \mathbf{0} & -\mathbf{S}(\mathbf{v}_1(t)) \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & -\mathbf{S}(\mathbf{v}_N(t)) \\ \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

The expression of $\mathbf{K}_1(t)$ is not presented here as it is evident from the context and it is not required in the sequel. It remains to show that the pair $(\mathbf{A}_1(t), \mathbf{C}_1)$ is uniformly completely observable, i.e., that there exist positive constants ϵ_1, ϵ_2, and δ such that

$$\epsilon_1 \mathbf{I} \preceq \boldsymbol{\mathcal{W}}(t, t+\delta) \preceq \epsilon_2 \mathbf{I} \tag{8}$$

for all $t \geq t_0$, where $\boldsymbol{\mathcal{W}}(t_0, t_f)$ is the observability Gramian associated with the pair $(\mathbf{A}_1(t), \mathbf{C}_1)$ on $[t_0, t_f]$. Since the entries of both $\mathbf{A}_1(t)$ and $\mathbf{C}_1$ are continuous and bounded, the right side of (8) is evidently verified. Therefore, only the left side of (8) requires verification. Let

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}_1 \\ \vdots \\ \mathbf{d}_{N+1} \end{bmatrix} \in \mathbb{R}^{3(N+1)},\ \mathbf{d}_i \in \mathbb{R}^3,\ i = 1, \ldots, N+1,$$

be a unit vector and define

$$\mathbf{f}(\tau) := \begin{bmatrix} \sqrt{\alpha_1\beta_1}\left[\mathbf{d}_1 - \int_t^\tau \mathbf{S}(\mathbf{v}_1(\sigma))\,\mathbf{d}_{N+1} d\sigma\right] \\ \vdots \\ \sqrt{\alpha_N\beta_N}\left[\mathbf{d}_N - \int_t^\tau \mathbf{S}(\mathbf{v}_N(\sigma))\,\mathbf{d}_{N+1} d\sigma\right] \end{bmatrix} \in \mathbb{R}^{3N}.$$

It is easy to show that

$$\mathbf{d}^T \boldsymbol{\mathcal{W}}(t, t+\delta)\,\mathbf{d} = \int_t^{t+\delta} \|\mathbf{f}(\tau)\|^2 d\tau.$$

If there exists i, $1 \leq i \leq N$, such that $\mathbf{d}_i \neq \mathbf{0}$, then it is clear that $\|\mathbf{f}(t)\| = \lambda_1 > 0$ for all $t \geq t_0$. On the other hand, if $\mathbf{d}_i = \mathbf{0}$ for all $i = 1, \ldots, N$, then it must be $\|\mathbf{d}_{N+1}\| = 1$, $\mathbf{f}(t) = \mathbf{0}$, and

$$\left.\frac{\partial \mathbf{f}}{\partial \tau}(\tau)\right|_{\tau=t} = \begin{bmatrix} -\sqrt{\alpha_1\beta_1}\mathbf{S}(\mathbf{v}_1(t)) \\ \vdots \\ -\sqrt{\alpha_N\beta_N}\mathbf{S}(\mathbf{v}_N(t)) \end{bmatrix} \mathbf{d}_{N+1}.$$

Now, notice that, under Assumption 1, and from the definition of the vector observations, there exist i and j such that $\mathbf{v}_i(t)$ and $\mathbf{v}_j(t)$ are non-collinear for all t. Therefore, there exists $\lambda_2 > 0$ such that

$$\left\| \left.\frac{\partial \mathbf{f}}{\partial \tau}(\tau)\right|_{\tau=t} \right\| \geq \lambda_2.$$

In addition, the second derivative of $\mathbf{f}$ is bounded as the angular velocity is assumed bounded. Therefore, using Lemma 1, there exists $\delta_1 > 0$ and $\lambda_3 > 0$ such that $\|\mathbf{f}(t+\delta_1)\| \geq \lambda_3$ for all $t \geq t_0$. Therefore,

$$\underset{\substack{\lambda^*>0 \\ \delta^*>0}}{\exists} \ \underset{t \geq t_0}{\forall} \ \underset{\substack{\mathbf{d} \in \mathbb{R}^{3(N+1)} \\ \|\mathbf{d}\|=1}}{\forall} \ \|\mathbf{f}(t+\delta^*)\| \geq \lambda^*,$$

and, using Lemma 1 again,

$$\underset{\substack{\epsilon_1>0 \\ \delta>0}}{\exists} \ \underset{t \geq t_0}{\forall} \ \underset{\substack{\mathbf{d} \in \mathbb{R}^{3(N+1)} \\ \|\mathbf{d}\|=1}}{\forall} \ \mathbf{d}^T \boldsymbol{\mathcal{W}}(t, t+\delta)\,\mathbf{d} \geq \epsilon_1,$$

which completes the conditions for uniformly completely observable and therefore concludes the proof. ∎

B. Attitude observer

This section proposes an attitude observer assuming that the rate gyro bias is known. In addition, the following assumption is considered.

Assumption 2: The matrix $[\mathbf{r}_1 \ldots \mathbf{r}_N] \in \mathbb{R}^{3\times 3N}$ has full rank.

Remark 1: It is important to stress that, given a set of reference vectors (and corresponding vector observations) that satisfy Assumption 1, it is always possible to construct a set of reference vectors (and corresponding vector observations) such that Assumption 2 is satisfied. Indeed, let $\mathbf{r}_i \in \mathbb{R}^3$ and $\mathbf{r}_j \in \mathbb{R}^3$ denote two non-collinear reference vectors. Then, notice that the set of reference vectors $\{\mathbf{r}_1, \ldots, \mathbf{r}_N, \mathbf{r}_i \times \mathbf{r}_j\}$ satisfies Assumption 2, to which corresponds the set of vector observations $\{\mathbf{v}_1(t), \ldots, \mathbf{v}_N(t), \mathbf{v}_i(t) \times \mathbf{v}_j(t)\}$. Therefore, Assumption 2 does not impose, in practice, any conservativeness whatsoever.

In order to simplify the derivation of the attitude observer and the corresponding proofs, consider a column representation of the rotation matrix $\mathbf{R}(t)$ given by

$$\boldsymbol{x}_2(t) = \begin{bmatrix} \mathbf{z}_1(t) \\ \mathbf{z}_2(t) \\ \mathbf{z}_3(t) \end{bmatrix} \in \mathbb{R}^9,$$

where

$$\mathbf{R}(t) = \begin{bmatrix} \mathbf{z}_1^T(t) \\ \mathbf{z}_2^T(t) \\ \mathbf{z}_3^T(t) \end{bmatrix},\ \mathbf{z}_i(t) \in \mathbb{R}^3,\ i = 1, \ldots, 3.$$

It is straightforward to show that

$$\dot{\boldsymbol{x}}_2(t) = -\mathbf{S}_3(\boldsymbol{\omega}_m(t) - \mathbf{b}_{\boldsymbol{\omega}}(t))\,\boldsymbol{x}_2(t),$$

where

$$\mathbf{S}_3(\mathbf{x}) := \operatorname{diag}(\mathbf{S}(\mathbf{x}), \mathbf{S}(\mathbf{x}), \mathbf{S}(\mathbf{x})) \in \mathbb{R}^{9\times 9},\ \mathbf{x} \in \mathbb{R}^3.$$

From (2) it is possible to write the vector observations as a function of the column representation of the rotation matrix, as given by

$$\mathbf{v}(t) = \mathbf{C}_2 \boldsymbol{x}_2(t),$$

where

$$\mathbf{v}(t) = \begin{bmatrix} \mathbf{v}_1(t) \\ \vdots \\ \mathbf{v}_N(t) \end{bmatrix} \in \mathbb{R}^{3N}$$

and

$$\mathbf{C}_2 = \begin{bmatrix} r_{11} & 0 & 0 & r_{12} & 0 & 0 & r_{13} & 0 & 0 \\ 0 & r_{11} & 0 & 0 & r_{12} & 0 & 0 & r_{13} & 0 \\ 0 & 0 & r_{11} & 0 & 0 & r_{12} & 0 & 0 & r_{13} \\ & & & & \vdots & & & & \\ r_{N1} & 0 & 0 & r_{N2} & 0 & 0 & r_{N3} & 0 & 0 \\ 0 & r_{N1} & 0 & 0 & r_{N2} & 0 & 0 & r_{N3} & 0 \\ 0 & 0 & r_{N1} & 0 & 0 & r_{N2} & 0 & 0 & r_{N3} \end{bmatrix},$$

$\mathbf{C}_2 \in \mathbb{R}^{3N\times 9}$, where $\mathbf{r}_i = \begin{bmatrix} r_{i1} & r_{i2} & r_{i3} \end{bmatrix}^T \in \mathbb{R}^3$. Notice that, under Assumption 2, matrix $\mathbf{C}_2$ has full rank.

Consider the attitude observer given by

$$\begin{aligned}\dot{\hat{\boldsymbol{x}}}_2(t) &= -\mathbf{S}_3\left(\boldsymbol{\omega}_m(t) - \mathbf{b}_{\boldsymbol{\omega}}(t)\right)\hat{\boldsymbol{x}}_2(t) \\ &\quad +\mathbf{C}_2^T\mathbf{Q}^{-1}\left[\mathbf{v}(t) - \mathbf{C}_2\hat{\boldsymbol{x}}_2(t)\right],\end{aligned} \tag{9}$$

where $\mathbf{Q} = \mathbf{Q}^T \in \mathbb{R}^{3N\times 3N}$ is a positive definite matrix, and define the error variable $\tilde{\boldsymbol{x}}_2(t) = \boldsymbol{x}_2(t) - \hat{\boldsymbol{x}}_2(t)$. Then, the observer error dynamics are given by

$$\dot{\tilde{\boldsymbol{x}}}_2(t) = \boldsymbol{\mathcal{A}}_2(t)\tilde{\boldsymbol{x}}_2(t), \tag{10}$$

where $\boldsymbol{\mathcal{A}}_2(t) := -\left[\mathbf{S}_3\left(\boldsymbol{\omega}_m(t) - \mathbf{b}_{\boldsymbol{\omega}}(t)\right) + \mathbf{C}_2^T\mathbf{Q}^{-1}\mathbf{C}_2\right]$.

The following theorem is the main result of this section.

Theorem 2: Suppose that the rate gyro bias is known and consider the attitude observer (9), where $\mathbf{Q} \succ \mathbf{0}$ is a design parameter. Then, under Assumption 2, the origin of the observer error dynamics (10) is a globally exponentially stable equilibrium point.

Proof: Consider the Lyapunov candidate function

$$V_2(t) := \frac{1}{2}\left\|\tilde{\boldsymbol{x}}_2(t)\right\|^2.$$

It is straightforward to show that

$$\dot{V}_2(t) = -\tilde{\boldsymbol{x}}_2^T(t)\mathbf{C}_2^T\boldsymbol{Q}^{-1}\mathbf{C}_2\tilde{\boldsymbol{x}}_2(t).$$

Now, as $\mathbf{C}_2$ is a constant matrix with full rank and $\mathbf{Q}$ is a positive definite matrix, it follows that $\mathbf{C}_2^T\mathbf{Q}^{-1}\mathbf{C}_2 \succ \mathbf{0}$. Therefore,

$$\dot{V}_2(t) \leq -\lambda_{min}\left(\mathbf{C}_2^T\mathbf{Q}^{-1}\mathbf{C}_2\right)\left\|\tilde{\boldsymbol{x}}_2(t)\right\|^2,$$

where $\lambda_{min}(\mathbf{X})$ corresponds to the minimum eigenvalue of matrix $\mathbf{X}$. This concludes the proof, see [16, Theorem 4.10]. ■

C. Cascade observer

This section presents the overall cascade observer and its stability analysis. In Section III-A an observer was derived, based directly on the vector observations, that provides an estimate of the bias, with globally exponentially stable error dynamics. The idea of the cascade observer is to feed the attitude observer proposed in Section III-B with the bias estimate provided by the bias observer proposed in Section III-A. The final nonlinear cascade observer reads as

$$\begin{cases}\dot{\hat{\mathbf{v}}}_1(t) = -\mathbf{S}\left(\boldsymbol{\omega}_m(t)\right)\hat{\mathbf{v}}_1(t) - \mathbf{S}\left(\mathbf{v}_1(t)\right)\hat{\mathbf{b}}_{\boldsymbol{\omega}}(t) + \alpha_1\tilde{\mathbf{v}}_1(t) \\ \quad\vdots \\ \dot{\hat{\mathbf{v}}}_N(t) = -\mathbf{S}\left(\boldsymbol{\omega}_m(t)\right)\hat{\mathbf{v}}_N(t) - \mathbf{S}\left(\mathbf{v}_N(t)\right)\hat{\mathbf{b}}_{\boldsymbol{\omega}}(t) + \alpha_N\tilde{\mathbf{v}}_N(t) \\ \dot{\hat{\mathbf{b}}}_{\boldsymbol{\omega}}(t) = \sum_{i=1}^N \beta_i\mathbf{S}\left(\mathbf{v}_i(t)\right)\tilde{\mathbf{v}}_i(t) \\ \dot{\hat{\boldsymbol{x}}}_2(t) = -\mathbf{S}_3\left(\boldsymbol{\omega}_m(t) - \hat{\mathbf{b}}_{\boldsymbol{\omega}}(t)\right)\hat{\boldsymbol{x}}_2(t) \\ \qquad\qquad +\mathbf{C}_2^T\mathbf{Q}^{-1}\left[\mathbf{v}(t) - \mathbf{C}_2\hat{\boldsymbol{x}}_2(t)\right]\end{cases}. \tag{11}$$

The error dynamics corresponding to the bias observer are the same and therefore Theorem 1 applies. Evidently, the use of an estimate of the bias instead of the bias itself in the attitude observer introduces an error, and the stability of the system must be further examined. In this situation, the error dynamics of the cascade observer can be written as

$$\begin{cases}\dot{\tilde{\boldsymbol{x}}}_1(t) = \boldsymbol{\mathcal{A}}_1(t)\tilde{\boldsymbol{x}}_1(t) \\ \dot{\tilde{\boldsymbol{x}}}_2(t) = \left[\boldsymbol{\mathcal{A}}_2(t) - \mathbf{S}_3\left(\tilde{\mathbf{b}}_{\boldsymbol{\omega}}(t)\right)\right]\tilde{\boldsymbol{x}}_2(t) + \mathbf{u}_2(t),\end{cases} \tag{12}$$

where $\mathbf{u}_2(t) := \mathbf{S}_3\left(\tilde{\mathbf{b}}_{\boldsymbol{\omega}}(t)\right)\boldsymbol{x}_2(t)$.

The following theorem is the main result of the paper.

Theorem 3: Consider the cascade attitude observer (11). Then, in the conditions of Theorem 1 and Theorem 2, the origin of the observer error dynamics (12) is a globally exponentially stable equilibrium point.

Proof: That $\tilde{\boldsymbol{x}}_1 = \mathbf{0}$ is a globally exponentially stable equilibrium point follows directly from Theorem 1. Next, it is shown that $\tilde{\boldsymbol{x}}_2 = \mathbf{0}$ is a globally exponentially stable equilibrium point of (12). First, consider the perturbed system

$$\dot{\tilde{\boldsymbol{x}}}_2(t) = \left[\boldsymbol{\mathcal{A}}_2(t) - \mathbf{S}_3\left(\tilde{\mathbf{b}}_{\boldsymbol{\omega}}(t)\right)\right]\tilde{\boldsymbol{x}}_2(t). \tag{13}$$

From Theorem 1 it follows that

$$\lim_{t\to 0}\left\|\mathbf{S}_3\left(\tilde{\mathbf{b}}_{\boldsymbol{\omega}}(t)\right)\right\| = 0.$$

Moreover, from Theorem 2, the origin of the undisturbed system $\dot{\tilde{\boldsymbol{x}}}_2(t) = \boldsymbol{\mathcal{A}}_2(t)\tilde{\boldsymbol{x}}_2(t)$ is a globally exponentially stable equilibrium point. Therefore, it is possible to conclude that the origin of the perturbed system (13) is a globally exponentially stable equilibrium point, see [16, Example 9.6]. Now, notice that, since $\mathbf{x}_2(t)$ corresponds to a column representation of the rotation matrix, which is norm-bounded, and $\tilde{\mathbf{b}}_{\boldsymbol{\omega}}(t)$ converges globally exponentially fast to zero, it follows that $\mathbf{u}_2(t)$ converges globally exponentially fast to zero. Therefore, the dynamics of $\tilde{\boldsymbol{x}}_2(t)$ correspond to those of a GES linear system driven by an exponentially decaying disturbance, from which follows that $\tilde{\boldsymbol{x}}_2 = \mathbf{0}$ is a globally exponentially stable equilibrium point, therefore concluding the proof. ■

D. Solution refinements

1) Full cascade observer: In the cascade observer proposed in Section III-C, the attitude observer considers only the bias estimate provided by the bias observer and the vector estimates are disregarded. Indeed, the observer employs, for feedback purposes, the output $\mathbf{v}(t)$ instead of the estimate $\hat{\mathbf{v}}(t)$. For performance purposes, particularly in the presence of sensor noise, it may be better to employ the vector estimate $\hat{\mathbf{v}}(t)$ provided by the bias observer. It should be stressed that the nominal asymptotic stability analysis is not affected. Indeed, for the full cascade observer

$$\begin{cases}\dot{\hat{\mathbf{v}}}_1(t) = -\mathbf{S}\left(\boldsymbol{\omega}_m(t)\right)\hat{\mathbf{v}}_1(t) - \mathbf{S}\left(\mathbf{v}_1(t)\right)\hat{\mathbf{b}}_{\boldsymbol{\omega}}(t) + \alpha_1\tilde{\mathbf{v}}_1(t) \\ \quad\vdots \\ \dot{\hat{\mathbf{v}}}_N(t) = -\mathbf{S}\left(\boldsymbol{\omega}_m(t)\right)\hat{\mathbf{v}}_N(t) - \mathbf{S}\left(\mathbf{v}_N(t)\right)\hat{\mathbf{b}}_{\boldsymbol{\omega}}(t) + \alpha_N\tilde{\mathbf{v}}_N(t) \\ \dot{\hat{\mathbf{b}}}_{\boldsymbol{\omega}}(t) = \sum_{i=1}^N \beta_i\mathbf{S}\left(\mathbf{v}_i(t)\right)\tilde{\mathbf{v}}_i(t) \\ \dot{\hat{\boldsymbol{x}}}_2(t) = -\mathbf{S}_3\left(\boldsymbol{\omega}_m(t) - \hat{\mathbf{b}}_{\boldsymbol{\omega}}(t)\right)\hat{\boldsymbol{x}}_2(t) \\ \qquad\qquad +\mathbf{C}_2^T\mathbf{Q}^{-1}\left[\hat{\mathbf{v}}(t) - \mathbf{C}_2\hat{\boldsymbol{x}}_2(t)\right]\end{cases},$$

the error dynamics are similar to (12), but with

$$\mathbf{u}_2(t) = \mathbf{S}_3\left(\tilde{\mathbf{b}}_{\boldsymbol{\omega}}(t)\right)\boldsymbol{x}_2(t) + \mathbf{C}_2^T\mathbf{Q}^{-1}\tilde{\mathbf{v}}(t).$$

Evidently, the steps of Theorem 3 apply yielding the same properties, as the additional term $\mathbf{C}_2^T\mathbf{Q}^{-1}\tilde{\mathbf{v}}(t)$ converges globally exponentially fast to zero.

2) Orthogonalization step: The cascade observer proposed in the paper yields an estimate of the rotation matrix $\mathbf{R}(t)$ given by

$$\hat{\mathbf{R}}(t) = \begin{bmatrix} \hat{\mathbf{z}}_1^T(t) \\ \hat{\mathbf{z}}_2^T(t) \\ \hat{\mathbf{z}}_3^T(t) \end{bmatrix}, \ \hat{\mathbf{z}}_i(t) \in \mathbb{R}^3, \ i = 1, \ldots, 3,$$

where

$$\hat{\boldsymbol{x}}_2(t) = \begin{bmatrix} \hat{\mathbf{z}}_1(t) \\ \hat{\mathbf{z}}_2(t) \\ \hat{\mathbf{z}}_3(t) \end{bmatrix} \in \mathbb{R}^9.$$

However, the estimate of the rotation matrix, $\hat{\mathbf{R}}(t)$, is not necessarily a rotation matrix as there is nothing in the observer structure imposing the restriction $\hat{\mathbf{R}}(t) \in SO(3)$. In fact, if this restriction is imposed, it is actually impossible to achieve global asymptotic stabilization due to topological limitations, see [15]. Nevertheless, the estimation error of the proposed observer converges globally exponentially fast to zero and therefore the corresponding rotation matrix restrictions are verified asymptotically.

In practice, both the vector observations and the rate gyro readings are subject to noise, which induces errors in the rotation matrix estimate not related to the initial transients that appear due to possible mismatch of initial conditions. Therefore, in order to obtain a better estimate, in the form of a rotation matrix, it is convenient to compute an orthogonal matrix that approximates the estimate provided by the cascade observer. Since, in practice, the estimate of the rotation matrix is very close (in steady-state) to an element of the Special Orthogonal Group $SO(3)$, one orthogonalization cycle suffices, as given by

$$\hat{\mathbf{R}}_f(t) = \frac{1}{2}\left(\hat{\mathbf{R}}(t) + \left[\hat{\mathbf{R}}^T(t)\right]^{-1}\right),$$

see [18]. Nevertheless, additional orthogonalization cycles may be employed should it be required in order to achieve solutions closer to $SO(3)$. Experimental results reveal that with two cycles the orthogonality error is of the same magnitude of the computational accuracy of low-cost hardware (below 10^{-12}). The projection of the estimate on $SO(3)$ is an alternative to the orthogonalization cycles. Although more expensive, it does provide solutions explicitly on $SO(3)$.

During the initial transients, which typically last less then 10 seconds, it may happen that the previous solution is not well-defined. In this case, the attitude may be simply obtained from the solution of the Wahba's problem as in traditional solutions resorting directly to the vector observations.

IV. Experimental results

In order to evaluate the proposed solutions in real world applications experiments were carried out as described in [19], where a high precision Motion Rate Table, Model 2103HT from Ideal Aerosmith [20], was employed that allows for accurate and reliable motion control and yields ground truth data for performance evaluation purposes. The table outputs the angular position of the table with a resolution of $0.00025\,°$. The IMU that was employed is the nanoIMU NA02-0150F50 [21], from MEMSENSE, which outputs data at a rate of 150 Hz. This 9 degree-of-freedom (DOF) Micro-Electro-Mechanical System (MEMS) device is a miniature, light weight, 3-D digital output sensor (it outputs acceleration, angular rate, and magnetic field data) featuring RS422 or I2C protocols, with built-in bias, sensitivity, and temperature compensation. The standard deviations of the noise of the outputs of the IMU are $0.95\,°/s$ for the angular velocity, $0.008\,\text{m/s}^2$ for the gravity acceleration, and 0.0015G for the magnetic field measurements.

Unfortunately, the calibration table distorts the magnetic field in the neighborhood of the IMU, even though it was attempted to place the IMU as far as possible from the other components of the experimental setup, by means of a small nonmagnetic bar, which elevates the sensor from the table top. Therefore, magnetic field measurements were simulated in the loop. Sensor noise was naturally added so that the results are as realistic as possible.

The motion rate table has three rotational joints which allow for movement about 3 orthogonally mounted axis, so called inner, middle, and outer axis, and that were defined as the x, y, and z axis of the body-fixed reference frame, so that the rotation from body-fixed coordinates to inertial coordinates is given by

$$\mathbf{R}(t) = \mathbf{R}_z\left(\theta_{out}(t)\right)\mathbf{R}_y\left(\theta_{mid}(t)\right)\mathbf{R}_x\left(\theta_{inn}(t)\right),$$

where $\mathbf{R}_x(.)$, $\mathbf{R}_y(.)$, and $\mathbf{R}_z(.)$ are the rotation matrices about the x, y, and z axis, respectively, and θ_{inn}, θ_{mid}, and θ_{out} are the inner, middle, and outer axis angles, respectively. The evolution of the inner, middle, and outer angles is depicted in Fig. 1. Notice that the angular motion full range is used, and if Euler angles were employed problems would have appeared due to singularities. Also, note that the angular velocity $\boldsymbol{\omega}(t)$, which is shown in Fig. 2, reaches interesting values, typical of many autonomous vehicles such as Autonomous Underwater Vehicles, Autonomous Ground Vehicles, or Unmanned Air Vehicles.

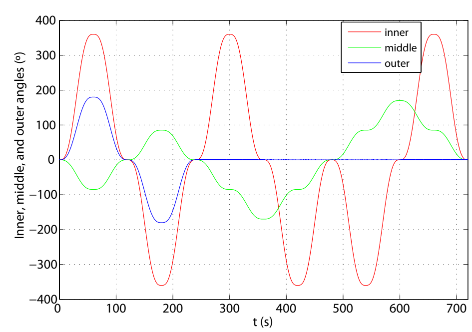

Fig. 1. Evolution of the inner, middle, and outer angles

The bias observer parameters were chosen as $\alpha_1 = \frac{9.8}{0.008}10^{-3}$, $\alpha_2 = \frac{0.5}{0.0015}10^{-3}$, and $\beta_1 = \beta_2 = 10^{-3}$, which are related to the norm of the vector observations and the noise of the sensors. The attitude observer parameter was chosen as $\mathbf{Q} = 0.25\mathbf{I}$ and all the initial estimates were set according to the first set of measurements. The initial bias estimate was set to zero.

The convergence rate of the observer is very fast and the steady-state is achieved in less than 1 s. The initial

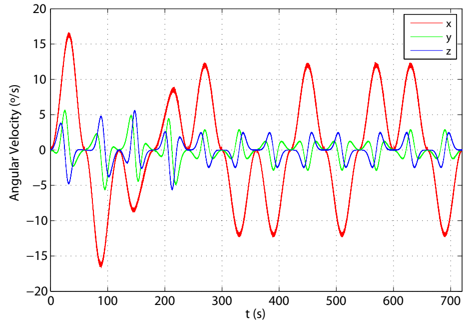

Fig. 2. Evolution of the angular velocity $\boldsymbol{\omega}(t)$

evolution of the error variables is not shown due to the lack of space. Using the Euler angle-axis representation for the rotation estimation error, the evolution of the angle error is shown in Fig. 3. The mean error is $0.18\,°$, which is a very good value considering the low-grade specifications of the IMU at hand. It is also comparable with the results obtained in simulation and it compares to a mean error of $0.13\,°$ for the solution proposed in [19]. However, the present solution is computationally efficient. Indeed, it does not require the online solution of Riccati equations nor the Wahba's problem, while the solution proposed in [19] requires all of these.

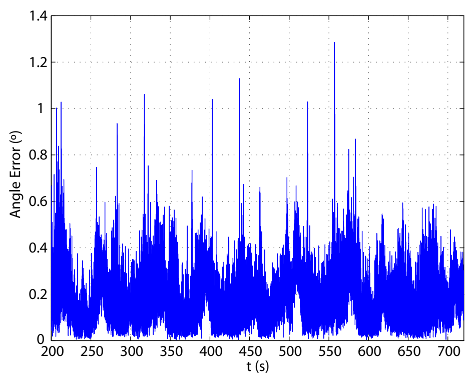

Fig. 3. Evolution of the angle error $\tilde{\theta}(t)$

V. Conclusions

This paper presented the design, analysis, and performance evaluation of a novel cascade observer for attitude estimation. The proposed solution resorts directly to a set of vector observations, in body-fixed coordinates, of known constant reference vectors in inertial coordinates, in addition to rate gyro readings. In short, a bias observer with GES error dynamics feeds a second attitude observer, which preserves GES error dynamics. An estimate of the rotation matrix is directly obtained, without the explicit solution of the Wahba's problem, and the observer gains do not require the solution of any differential equation. Therefore, the proposed system is computationally efficient and appropriate for application in platforms where computational resources are scarce. Furthermore, the present solution does not exhibit drawbacks common to attitude estimation solutions such as singularities, unwinding phenomena, or topological limitations for achieving global asymptotic stability. Compelling experimental results were provided with a a low-cost, low-power IMU, where a Motion Rate Table was employed for performance evaluation purposes, providing ground truth data.

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